

Hydrodynamics of the Chiral Dirac Spectrum

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(Dated: June 30, 2015)

We derive a hydrodynamical description of the eigenvalues of the chiral Dirac spectrum in the vacuum and in the large N (volume) limit. The linearized hydrodynamics supports sound waves. The stochastic relaxation of the eigenvalues is captured by a hydrodynamical instanton configuration which follows from a pertinent form of Euler equation. The relaxation from a phase of localized eigenvalues and unbroken chiral symmetry to a phase of de-localized eigenvalues and broken chiral symmetry occurs over a time set by the speed of sound. We show that the time is $\Delta\tau = \pi\rho(0)/2\beta N$ with $\rho(0)$ the spectral density at zero virtuality and $\beta = 1, 2, 4$ for the three Dyson ensembles that characterize QCD with different quark representations in the ergodic regime.

PACS numbers: 12.38Aw, 12.38Mh, 71.10Pm

1. Introduction. QCD with light quarks breaks spontaneously chiral symmetry. As a result, the light quarks transmute to massive constituents which make up for most of the visible mass in our universe. Empirical evidence for the spontaneous breaking of chiral symmetry is in the form of light pions and kaons in nature [1]. Dedicated first principle QCD lattice simulations with light quarks have established that chiral symmetry is spontaneously broken with a finite chiral condensate and an octet of light mesons [2].

The spontaneous breaking of chiral symmetry is characterized by a large accumulation of eigenvalues of the Euclidean Dirac operator near zero virtuality [3]. This phenomenon signals the onset of an ergodic regime in the chiral Dirac spectrum. Other regimes where the light quarks diffuse or undergo ballistic motion can also be identified [4]. In many ways light quarks interacting via colored Yang-Mills fields behave like disordered electrons in metallic grains.

The essentials of the ergodic regime are captured by a chiral random matrix model [5]. In short, the QCD Dirac spectrum near zero virtuality only retains that the QCD Dirac operator is chiral or block off-diagonal, with random entries that are sampled from the three universal Dyson ensembles [6] with a Gaussian weight thanks to the central limit theorem. The spectrum corrections are also generic and follow from the neighboring diffusive regime with the light quark return probability falling like a power law in proper time [4]. Both regimes are separated by the Thouless energy [4, 7].

In this letter we develop a hydrodynamical description of the eigenvalues of the QCD Dirac operator in the diffusive regime [4], much along the lines of our recent studies of the eigenvalues of the Polyakov line at large number of colors [8]. We will use this derivation to obtain the following new results: 1/ A hydrostatic solution for the spectral density beyond Wigner semi-circular distribution; 2/ a hydrodynamical instanton that captures the stochastic relaxation of the eigenvalues of the QCD Dirac operator for low virtuality; 3/ a dynamical relaxation time for restoring and/or breaking spontaneously chiral symmetry directly from the QCD Dirac spectrum; 4/ an estimate of this time for the dyon liquid model.

2. Chiral Dirac spectrum. The original chiral random matrix model partition function for the eigenvalues of the chiral Dirac spectrum was initially suggested as a null dynamical assumption for the generic analysis of the macroscopic chiral moments in the instanton liquid model [9]. However and more importantly, it was noted that this assumption provides a universal description of the chiral moments in the microscopic limit [5, 10]. For QCD with N_f quarks of equal masses m in the complex representation [5, 9]

$$Z_2[m] = \int dT \prod_{f=1}^{N_f} \det(m^2 + T^\dagger T) e^{-\frac{a}{2} N \text{Tr}(T^\dagger T)} \quad (1)$$

Here T is a symmetric random $C^{N \times N}$ complex matrix capturing the random hopping between N -left and N -right zero modes. (1) was generalized to all Dyson ensembles with $\beta = 1, 2, 4$ corresponding to quarks in different representations in [6],

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$$Z_\beta[m] = \int \prod_{i < j}^N |\lambda_i^2 - \lambda_j^2|^\beta \prod_i^N \lambda_i^\alpha (\lambda_i^2 + m^2)^{N_f} e^{-\frac{a\beta N}{2} \lambda_i^2} d\lambda_i \quad (2)$$

with $\alpha = \beta(\nu + 1) - 1$. ν measures the topological asymmetry of T , that is the difference between the number of zero modes and anti-zero modes for instance. Here λ_i^2 are the eigenvalues of the squared Wishart matrix $W = T^\dagger T$. The overall normalization in (1-2) are omitted.

The Gaussian measure is generic at large N by the central limit theorem, with the parameter a fixed by the chiral condensate. In the microscopic limit whereby the Dirac spectrum near zero-virtuality is magnified so that $N\lambda_i = 1$, the interactions between the eigenvalues as mediated by the gauge-fields are chaotic. As a result only the chiral structure of the Dirac Hamiltonian and the symmetry of the matrix entries under time-reversal are relevant [11]. This is the universal regime of ergodicity shared by most disordered electronic systems in the mesoscopic limit [12].

Alternatively, (2) can be regarded as the normalization of the squared and real many-body wave-function

$$Z_\beta[m] = \int \prod_{i=1}^N d\lambda_i |\Psi_0[\lambda_i]|^2 \quad (3)$$

which is the zero-mode solution to the Shrodinger equation $H_0\Psi_0 = 0$ with the self-adjoint squared Hamiltonian

$$H_0 \equiv \sum_{i=1}^N (-\partial_i + \mathbf{a}_i)(\partial_i + \mathbf{a}_i) \quad (4)$$

with $\partial_i \equiv \partial/\partial\lambda_i$ and the pure gauge potential $\mathbf{a}_i \equiv \partial_i S$. The mass parameter is $1/2$. The Vandermond contribution $\Delta = \prod_{i < j} |\lambda_{ij}^2|^\beta$ induces a diverging 2-body part. It is convenient to re-absorb it using a similarity transformation by defining $\Psi = \Psi_0/\sqrt{\Delta}$ and the new Hamiltonian

$$H = (1/\sqrt{\Delta}) H_0 \sqrt{\Delta} \quad (5)$$

3. Hydrodynamics. We can use the collective coordinate method in [13] to re-write (5) in terms of the density of eigenvalues as a collective variable $\rho(\lambda) = \sum_{i=1}^N \delta(\lambda - \lambda_i)$. After some algebra we obtain

$$H = \int d\lambda (\partial_\lambda \pi \rho \partial_\lambda \pi + \rho \mathbf{u}[\rho]) \quad (6)$$

with the potential-like contribution $\mathbf{u}[\rho] \equiv \mathbf{A}^2$

$$\begin{aligned} \mathbf{A} = & \frac{a\beta N}{2} \lambda - \frac{\beta\pi}{2} [\rho_H(\lambda) - \rho_H(-\lambda)] \\ & + \partial_\lambda \ln \sqrt{\rho} - \frac{\alpha}{2\lambda} - \frac{N_f \lambda}{\lambda^2 + m^2} \end{aligned} \quad (7)$$

Here ρ_H is the Hilbert transform of ρ

$$[\rho]_H \equiv \rho_H(\lambda) = \frac{1}{\pi} \mathcal{P} \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} \quad (8)$$

The canonical pair, $\pi(\lambda)$ and $\rho(\lambda)$ satisfies the Poisson bracket $\{\pi(\lambda), \rho(\lambda')\} = \delta(\lambda - \lambda')$. We identify the collective fluid velocity with $v = \partial_\lambda \pi$ and re-write (6) in the more familiar hydrodynamical form

$$H \approx \int d\lambda \rho(\lambda) (v^2 + \mathbf{u}[\rho]) \approx \int d\theta \rho(\lambda) |v + i\mathbf{A}|^2 \quad (9)$$

ignoring ultra-local terms. The equation of motion for ρ yields the current conservation law $\partial_t \rho = -2\partial_\lambda (\rho v)$, and the equation of motion for v gives the Euler equation

$$\begin{aligned} \partial_t v = \{H, v\} = & -\partial_\lambda [v^2 + \mathbf{A}^2 - \partial_\lambda \mathbf{A} - \mathbf{A} \partial_\lambda \ln \rho \\ & + \pi \beta [\mathbf{A} \rho]_H(\lambda) + \pi \beta [\mathbf{A} \rho]_H(-\lambda)] \end{aligned} \quad (10)$$

All the relations hold for large but finite N , to allow for a smoothening of the density.

4. Hydrostatic solution. The hydrostatic solution corresponds to the minimum of (9) with $v = 0$ and thus $\mathbf{A}(\lambda) = 0$. In terms of the chirally symmetric combination $\rho^\chi(\lambda) \equiv \rho(\lambda) + \rho(-\lambda)$ we have

$$\begin{aligned} 0 = & \frac{a\beta N}{2} \lambda - \frac{\beta\pi}{2} \rho^\chi_H(\lambda) \\ & + \frac{1}{2} \partial_\lambda \ln \rho^\chi - \frac{\alpha}{2\lambda} - \frac{N_f \lambda}{\lambda^2 + m^2} \end{aligned} \quad (11)$$

In the large N limit only the terms in the first line survive with

$$\rho_0^\chi(\lambda) = \frac{Na}{\pi} \sqrt{\frac{4}{a} - \lambda^2} \quad (12)$$

which is Wigner Semi-circular distribution within the support $|\lambda| \leq 2/\sqrt{a}$. To correct it in $1/N$ we define $\rho^\chi \approx \rho_0^\chi + \rho_1$ subject to the condition $\int \rho_1 d\lambda = 0$, so that

$$-\beta\pi \rho_1(\lambda) = \frac{1}{\lambda} (\alpha + 2N_f) - \frac{\lambda a}{4 - a\lambda^2} \quad (13)$$

The closed form solution is

$$\begin{aligned} \rho_1(\lambda) = & -\frac{\alpha + 2N_f}{\beta} \delta(\lambda) + \frac{|\lambda|}{\beta \sqrt{4/a}} \delta(\lambda^2 - 4/a) \\ & + \frac{\alpha + 2N_f - \frac{1}{2\sqrt{4/a}}}{\beta \pi \sqrt{4/a - \lambda^2}} \end{aligned} \quad (14)$$

with the support unchanged. We note that (14) shows a delta-function accumulation at zero virtuality with the negative strength $\alpha + 2N_f$.

The general solution to (11) for large but finite N can be sought in the massless case or $m = 0$, by multiplying (11) by $2\rho^x \lambda(\lambda^2 + m^2)$ and taking the Hilbert transform. The result is

$$a\beta N \left(\lambda^2 \rho_H^x - \lambda \frac{2N}{\pi} \right) - \frac{\beta\pi}{2} \lambda \left(\rho_H^x{}^2 - \rho^x{}^2 \right) + \lambda \partial_\lambda \rho_H^x - (\alpha + 2N_f) \rho_H^x = 0 \quad (15)$$

To solve it consider the extension of the resolvent

$$G(z) = \int d\lambda \frac{\rho(\lambda)}{z - \lambda} \quad (16)$$

to the upper and lower complex plane,

$$G^\pm(z \rightarrow \lambda) = \pi \rho_H^x(\lambda) \mp i\pi \rho^x(\lambda) \quad (17)$$

so that (15) now reads

$$a\beta N z^2 G - \frac{1}{2} \beta z G^2 + z \partial_z G - (\alpha + 2N_f) G - 2a\beta N^2 z = 0 \quad (18)$$

Defining $G(z) \equiv -(2/\beta) \partial_z \ln f(z)$ and setting $z^2 = -w$ and $f(\sqrt{-w}) = g(w)$, we have

$$2w \partial_w g + (-a\beta N w + 1 - \alpha - 2N_f) \partial_w g - \frac{1}{2} a\beta^2 N^2 g = 0 \quad (19)$$

We note that (19) is Laguerre-like except for the wrong sign in the last contribution. The general solution is a hyper-geometric function. The spectral density follows from the discontinuities of the solution using (17).

5. Dyson Coulomb gas. We note that (2) can be re-written in terms of 1-dimensional Dyson Coulomb gas. At large N the ensemble described by (1) is sufficiently dense to allow the change in the measure,

$$\prod_{i=1}^N d\lambda_i \approx e^{\mathcal{S}[\rho]} D\rho \quad (20)$$

with $\mathcal{S}[\rho] = \int d\lambda \rho(\lambda) \ln(\rho_0/\rho(\lambda))$ the Boltzmann entropy [14]. Thus

$$Z_\beta[m] \rightarrow \int D\rho e^{-\Gamma[\beta, m; \rho]} \quad (21)$$

with the effective action

$$\begin{aligned} \Gamma[\beta, m; \rho] = & \int d\lambda \rho(\lambda) \left(\frac{a\beta N}{2} \lambda^2 - \alpha \ln(\lambda) - N_f \ln(\lambda^2 + m^2) \right) \\ & - \frac{\beta}{2} \int d\lambda d\lambda' \rho(\lambda) \rho(\lambda') \ln(\lambda^2 - \lambda'^2) \\ & - \left(\frac{\beta}{2} - 1 \right) \int d\lambda \rho \ln \rho \end{aligned} \quad (22)$$

The β contribution is the self Coulomb subtraction and is consistent with the subtraction in the Hilbert transform. The saddle point equation $\delta\Gamma/\delta\rho = 0$ yields the hydrostatic equation (11) using the symmetric density ρ^x .

6. Hydrodynamical instanton. Following the initial observation in [8], we note that the fixed time zero energy solution to (9) is an instanton with imaginary velocity $v = -i\mathbf{A}$. The conserved current $j \equiv \rho v$ satisfies ($\tau = it$)

$$\begin{aligned} \partial_\tau \rho(\lambda) = \partial_\lambda [a\beta N \lambda \rho(\lambda) - \beta \pi \rho(\lambda) [\rho_H(\lambda) - \rho_H(-\lambda)] \\ + \partial_\lambda \rho(\lambda) - \frac{\alpha}{\lambda} \rho(\lambda) - \frac{2N_f \lambda}{\lambda^2 + m^2} \rho(\lambda)] \end{aligned} \quad (23)$$

We identify τ with the stochastic time, and (23) describes the stochastic relaxation of the chiral eigenvalue density of the chiral Dirac spectrum (out of equilibrium) to its asymptotic (in equilibrium) hydro-static solution. For $m = 0$, (23) can be rewritten as

$$\begin{aligned} \partial_\tau \rho^x = a\beta N \partial_\lambda (\lambda \rho^x) - \beta \pi \partial_\lambda (\rho^x \rho_H^x) \\ + \partial_{\lambda\lambda} \rho^x - (\alpha + 2N_f) \partial_\lambda \left(\frac{1}{\lambda} \rho^x \right) \end{aligned} \quad (24)$$

After multiplying this equation by λ^2 , we can take its Hilbert transform. The result is

$$\begin{aligned} \partial_\tau \rho_H^x = a\beta N \partial_\lambda (\lambda \rho_H^x) - \frac{\beta\pi}{2} \partial_\lambda (\rho_H^x{}^2 - \rho^x{}^2) \\ + \partial_{\lambda\lambda} \rho_H^x - (\alpha + 2N_f) \partial_\lambda \left(\frac{1}{\lambda} \rho_H^x \right) \end{aligned} \quad (25)$$

With the usual definition, this gives an equation for the time dependent Green's function

$$\begin{aligned} \partial_\tau G = a\beta N \partial_z (zG) - \frac{\beta}{2} \partial_z (G^2) \\ + \partial_{zz} G - (\alpha + 2N_f) \partial_z \left(\frac{1}{z} G \right) \end{aligned} \quad (26)$$

We note that the stationary solution fulfills

$$a\beta N z G - \frac{\beta}{2} G^2 + \partial_z G - (\alpha + 2N_f) \frac{1}{z} G + C = 0 \quad (27)$$

The constant C is fixed by noting that for large N and large z , $G/N \approx 2/z$ so that $C = 4aN^2$, in agreement with (18).

The general time-dependent solution can be analyzed by first re-scaling $\tau \rightarrow \tau/N$. In the large N limit G solves

$$\partial_\tau G + \beta \left(\frac{1}{N} G - az \right) \partial_z G - a\beta G = 0 \quad (28)$$

which is a Burgers like nonlinear PDE. We can solve it with the method of complex characteristics [15]. For that we introduce curves in the space of z and τ , parametrized by s and labeled by z_0 , along which the nonlinear PDE follows from ordinary ODE

$$\begin{aligned} \frac{dz}{ds} &= \beta \left(\frac{1}{N} G - az \right) \\ \frac{dG}{ds} &= a\beta G \\ \frac{d\tau}{ds} &= 1 \end{aligned} \quad (29)$$

with the condition that $z(\tau = 0) = z_0$ and $\tau(s = 0) = 0$ (which means $\tau = s$). These ODE can be solved for a specific initial condition.

7. Sound waves. To gain some insights to the general time-dependent solutions, we analyze first the hydrodynamical equations in the linearized density approximation. For that, it is convenient to identify the classical hydrodynamical action associated to (6). Using the standard canonical procedure we found

$$\mathbf{S} = \int dt d\lambda \rho(\lambda) (v^2 - \mathbf{u}[\rho]) \quad (30)$$

which is linearized by

$$\rho \approx \rho_0(\lambda) + 2\partial_\lambda \varphi \quad \text{and} \quad \rho v \approx -\partial_t \varphi \quad (31)$$

Inserting (31) into \mathbf{S} yields in the quadratic approximation

$$\mathbf{S}_2 = \int dt \frac{d\lambda}{\rho_0(\lambda)} ((\partial_t \varphi)^2 - \rho_0^2(\lambda) W^2[\varphi]) \quad (32)$$

with the potential

$$W[\varphi] = \beta\pi\partial_\lambda [\varphi_H(\lambda) + \varphi_H(-\lambda)] - \partial_\lambda \left(\frac{\partial_\lambda \varphi}{\rho_0(\lambda)} \right) \quad (33)$$

For $\rho(\lambda) = \rho_0(\lambda)$, after the rescaling $Nt \rightarrow t$, (32) simplifies for large N

$$\begin{aligned} \mathbf{S}_2 &\approx N^2 \int dt \frac{d\lambda}{\rho_0(\lambda)} \\ &\times \left((\partial_t \varphi)^2 - \frac{(\pi\beta\rho_0(\lambda))^2}{N^2} (\partial_\lambda \varphi_H(\lambda) + \partial_\lambda \varphi_H(-\lambda))^2 \right) \end{aligned} \quad (34)$$

The speed of sound $v_s(\lambda) = 2\pi\beta\rho_0(\lambda)/N$ is local in the chiral spectrum. It is $v_s(0) = 2\beta\sqrt{a}$ at zero virtuality. We note that (34) is extensive with N for $\rho_0(\lambda)/N$ normalized to 1.

8. Chiral relaxation time. An interesting dynamical question regarding the chiral Dirac spectrum is the typical relaxation time associated to the formation or disappearance of the spontaneous breaking of chiral symmetry. This issue is important for QCD with matter undergoing dynamical chiral symmetry restoration or breaking close to the chiral transition temperature. Since the set-up involves off-equilibrium QCD in a moderately strong coupling regime, first principle calculations are usually challenging.

Here we answer this question by focusing on the time it takes for the eigenvalues near zero-virtuality viewed as a hydrodynamical fluid to re-arrange stochastically. For that, consider that at $\tau = 0$ all the eigenvalues are localized at zero virtuality to mock up an initial and chirally restored phase. The relaxation of this phase to a spontaneously broken chiral phase is characterized by the time it takes the sound wave to fill up the gap or the so-called zero-mode-zone (ZMZ) spanned by Wigner semi-circle.

To show this we use the initial condition $G(z = z_0, \tau = 0) = 2N/z_0$ in (29) to obtain the solution

$$G = e^{a\beta\tau} \frac{2N}{z_0} \quad (35)$$

The remaining equation is now

$$\frac{dz}{d\tau} = \beta \left(\frac{2}{z_0} e^{a\beta\tau} - az \right) \quad (36)$$

which yields

$$z = \frac{1}{az_0} e^{a\beta\tau} + \left(z_0 - \frac{1}{az_0} \right) e^{-a\beta\tau} \quad (37)$$

Solving for z_0 and inserting the answer in (35) yield

$$\begin{aligned} \frac{G(z)}{N\sqrt{a}} &= \left(1 - e^{-v_s(0)\tau\sqrt{a}} \right)^{-1} \\ &\times \left(z\sqrt{a} \pm \sqrt{(z\sqrt{a})^2 - 4(1 - e^{-v_s(0)\tau\sqrt{a}})} \right) \end{aligned} \quad (38)$$

The spectral density follows from the discontinuity of (12)

$$\rho(\lambda) = \frac{aN}{\pi} \left(1 - e^{-v_s(0)\tau\sqrt{a}}\right)^{-1} \times \left(\frac{4}{a} \left(1 - e^{-v_s(0)\tau\sqrt{a}}\right) - \lambda^2\right)^{\frac{1}{2}} \quad (39)$$

which is seen to interpolate between a delta-function $N\delta(\lambda)$ at $\tau = 0$ and a Wigner semi-circle at asymptotic τ at a rate given by the speed of sound $v_s(0)$ at zero virtuality. The typical time in physical units is

$$\Delta\tau \equiv \frac{a}{v_s(0)} = \frac{\pi\rho(0)}{2\beta N} = \frac{|\langle q^\dagger q \rangle|}{2\beta \mathbf{n}} \quad (40)$$

Recall that in chiral random matrix theory the scale a is related to the chiral condensate by the Banks-Casher formula $V_4 \langle q^\dagger q \rangle = -\pi\rho(0) = -N\sqrt{a}$. Thus, the last equality with $\mathbf{n} = N/V_4$. Near zero virtuality and close to the ergodic regime this time is universal. It is non-perturbative and fully gauge-invariant.

In the QCD vacuum we may identify \mathbf{n} with the instanton density [1] (and references therein). Typically, $|\langle q^\dagger q \rangle| \approx 1/(1\text{ fm}^3)$ and $\mathbf{n} \approx 1/(1\text{ fm}^4)$ so that $\Delta\tau \approx 1\text{ fm}/4$ for QCD with $N_c = 3$ ($\beta = 2$), which is short. In matter, both the chiral condensate and the instanton density change. For a dyon liquid model in the confining phase in the range $0.5 < T/T_c < 1$, we may identify $\mathbf{n} = \mathbf{n}_D/N_c$ with \mathbf{n}_D the dyon density [16]. We recall that the instanton splits to N_c dyons with only the KK dyon carrying the zero-mode. From the analysis presented in [17] we have for $N_c = 2$ ($\beta = 1$) and $N_f = 1$

$$\Delta\tau = \frac{|\langle q^\dagger q \rangle|}{2\beta \mathbf{n}_D/N_c} \approx \frac{1.25}{T} \left(C \frac{e^{-\frac{\pi}{\alpha_s(T)}}}{\alpha_s^2(T)} \right)^{0.63} \quad (41)$$

with $\alpha_s(T) = (10/3) \ln(T/0.36T_c)$ the running coupling, and C a constant of order 1 in the dyon density which is fixed by lattice measurements.

9. Conclusions. The hydrodynamical description of the chiral Dirac spectrum captures some key aspects of the stochastic relaxation of the Dirac eigenvalues in the diffusive regime. The hydrodynamical set-up supports an instanton that describes the stochastic relaxation of the Dirac eigenvalues as a fluid. The fluid exhibits sound waves that can be used to estimate the time it takes for a localized density with chiral symmetry restored, to de-localize to a Wigner semi-circle with chiral symmetry restored. The relaxation time of a fluid of Dirac eigenvalues near zero virtuality captures a typical equilibration time for chiral symmetry breaking or restoration in QCD that is robust and gauge-independent.

Acknowledgements The work of YL and IZ is supported in part by the U.S. Department of Energy under Contracts No. DE-FG-88ER40388. The work of PW is supported by the DEC-2011/02/A/ST1/00119 grant and the UMO-2013/08/T/ST2/00105 ETIUDA scholarship of the (Polish) National Centre of Science. PW thanks Stony Brook University for its hospitality during the completion of this work.

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